11/29/23 <u>Hecke eigensheaves via Chiral homology</u> Speaker: Justin
Notes: Nikolay

<u>The plan:</u> 1^{st} goal: Complete BD construction:

$$\text{FF} + \text{localization} + \text{chiral homology} \Longrightarrow \mathrm{Fun}(\mathrm{Op}_{\check{G}}(X)) \to \Gamma(\mathrm{Bun}_G, D_{crit})$$

Want two properties:

i) quantizes Hitchin system (comp. w/ filtrations & assoc. graded is Hitchin map)

ii) compatibility w/ Satake

We'll focus on (ii).

$$\chi \in \mathrm{Op}_{\check{G}}(X)(k) \rightsquigarrow M_\chi := k_\chi \otimes_{\mathrm{Fun}(\mathrm{Op}_{\check{G}}(X))} \mathcal{D}_{crit} \in D_{crit}(\mathrm{Bun}_G)$$

(ii) $\Longrightarrow$ M_χ is Hecke eigensheaf.

We'll also sketch the construction of all cuspidal eigensheaves using FLE at critical level.

<u>§ The center of factorization algebra</u>

A = Fact. alg / X. A is object of $D(\mathrm{Ran}\, X)$ w/ some extra structure (fact. iso's & unital structure).

I = fin. set $\rightsquigarrow$ $A\text{-mod}^{\text{fact}}_{X^I} \circlearrowleft D(X^I)$.

Concretely, an object of $A\text{-mod}^{\text{fact}}_{X^I}$ consists of:

- an object of $D(\mathrm{Ran}_I)$, where
$$\mathrm{Ran}_I = \{(x_I, y) \mid x_I \in X^I,\ y \subseteq X \text{ fin, s.t. } \{x_I\} \subseteq y\}$$
- isomorphism $j^! u^! M \simeq j^!(A \boxtimes M)$, where
$$(\mathrm{Ran} \times \mathrm{Ran}_I)_{disj} \overset{j}{\hookrightarrow} \mathrm{Ran} \times \mathrm{Ran}_I \xrightarrow{u} \mathrm{Ran}_I$$

compatible w/ unital fact. structure on A + higher comps.

$X^I \hookrightarrow \mathrm{Ran}_I \quad \rightsquigarrow \quad A\text{-mod}^{\mathrm{fact}}_{X^I} \xrightarrow{\mathrm{oblv}_A} D(X^I)$

$x_I \longmapsto (x_I, \{x_i\}) \qquad\qquad M \longmapsto M|^!_{X^I}$

↖ "Forget the A-fact. structure" is conservative.

$D(\mathrm{Ran}) \curvearrowleft A\text{-mod}^{\mathrm{Fact}}_{\mathrm{Ran}} := \lim_I A\text{-mod}^{\mathrm{Fact}}_{X^I}$

$$\begin{tikzcd} A\text{-mod}^{\mathrm{Fact}}_{\mathrm{Ran}} \arrow[d, "\mathrm{oblv}_A"] \\ D(\mathrm{Ran}) \end{tikzcd}$$

E.g: $\mathrm{Vac}_A \in A\text{-mod}^{\mathrm{fact}}_{\mathrm{Ran}}$, $\mathrm{oblv}_A(\mathrm{Vac}_A) = A$

"A as a Fact. module over itself"

$$\boxed{Z(A) := \underline{\mathrm{End}}_{A\text{-mod}^{\mathrm{Fact}}_{\mathrm{Ran}} \circlearrowleft D(\mathrm{Ran})}(\mathrm{Vac}_A) \in D(\mathrm{Ran}) \quad \in \mathrm{AssocAlg},\ \mathrm{Fact\,Alg}.}$$

H^0 is commutative? $A =$ vacuum of KM, $H^0 =$ FF center

§ Commutative factorization algebras

$B =$ comm. Fact. alg

Prop B lifts to an object of $\mathrm{Com\,Alg}(\mathrm{Com\,Fac\,Alg})$.

Pf $\mathrm{Com\,Fact\,Alg} \xrightarrow[\sim]{\text{sym. mon.}} \mathrm{Com\,Alg}(D(X)) \simeq \mathrm{Com\,Alg}(\mathrm{Com\,Alg}(D(X)))$

$\Longrightarrow B \in \mathrm{Com\,Fact\,Alg} \simeq \mathrm{Com\,Alg}(\mathrm{Com\,Fact\,Alg})$

$\longrightarrow \mathrm{Assoc\,Alg}(\mathrm{Com\,Fact\,Alg})$

$\longrightarrow \mathrm{Assoc\,Alg}(\mathrm{Fact\,Alg})$

Prop $C_\nabla(X, B)$ is naturally a comm. alg.

Lem $C_\nabla(X, -) : \mathrm{Fact\,Alg} \longrightarrow \mathrm{Vect}$ is sym. mon.

pf sketch:

$$\begin{tikzcd} \mathrm{Fact\,Alg} \arrow[r, "\mathrm{oblv}"] \arrow[dr, "C_\nabla(X,-)"'] & D(\mathrm{Ran}) \arrow[d, "C_{dR}(\mathrm{Ran},-)"] \\ & \mathrm{Vect} \end{tikzcd}$$

oblv is symm. mon. $C_{dR}(\mathrm{Ran}, -)$ is apriori oplax symm. mon.

(right adjoint is $p^!$)

In fact, it is strictly symm. on unital sheaves.

(Rozenblyum's thesis "Connections on moduli spaces")

<u>Pf of prop</u> Combine lemma + previous prop

§ Local - to - global construction

A = Fact. alg, B = comm fact. alg.

w.r.t pointwise $\otimes$.

<u>Prop - construction</u> $B \longrightarrow Z(A)$ in Assoc. Alg (Fact Alg)

$$\rightsquigarrow C_\nabla(X, B) \curvearrowright C_\nabla(X, A)$$

<u>Pf</u> $B \to Z(A) \rightsquigarrow B \to A$ in $D(\mathrm{Ran})$

$$B \overset{!}{\otimes} A \longrightarrow A \overset{C_\nabla(X,-)}{\rightsquigarrow} C_\nabla(X, B \overset{!}{\otimes} A) \longrightarrow C_\nabla(X, A)$$

is

$$C_\nabla(X,B) \otimes C_\nabla(X,A)$$

<u>Ex</u> $A = A_{\mathfrak{g},\mathrm{crit}}$, $B = \mathrm{Fun}(\mathrm{Op}_{\check{G}}(D))$ $\quad B \to Z(A)$ isom on H^0

$$A_{\mathfrak{g}} = U^{ch}(\mathfrak{g} \otimes \mathcal{D}_X)$$

Chevalley-Eilenberg complex

$$\rightsquigarrow C_\nabla(X, A_{\mathfrak{g}}) = C(C_{dR}(X, \mathfrak{g} \otimes \mathcal{D}_X))$$

$$= C(R\Gamma(X, \mathfrak{g} \otimes \mathcal{O}_X))$$

$$= \mathcal{D}_{\mathrm{Bun}_G}\Big|^!_{P_G^{\mathrm{triv}}}$$

$$C_\nabla(X, A_{\mathfrak{g},\kappa}) = \mathcal{D}_{\mathrm{Bun}_{G,\kappa}}\Big|^!_{P_G^{\mathrm{triv}}}$$

$$C_\nabla(X, B) = \mathrm{Fun}(\mathrm{Op}_{\check{G}}(X)) \quad \text{(is theorem)}$$

Upgraded FF (no ref. yet?!)

$$\mathrm{Fun}(\mathrm{Op}_{G^\vee}(D)) \longrightarrow Z(A_{\mathfrak{g},\mathrm{crit}})$$

Prop/Constr. $\rightsquigarrow \mathrm{Fun}(\mathrm{Op}_{G^\vee}(X)) \curvearrowright C_\nabla(X, A_{\mathfrak{g},\mathrm{crit}}) = D_{\mathrm{Bun}_G,\mathrm{crit}}\big|^!_{p_G^{\mathrm{triv}}}$

Also want:

(i) Action of $\mathrm{Fun}(\mathrm{Op}_{G^\vee}(X))$ on entire $\mathcal{D}_{\mathrm{Bun}_G,\mathrm{crit}}$

(ii) comp. w/ Satake

Break

§ Localization & Chiral homology

Kevin constructed a functor

$$\mathrm{Loc}_{G,\kappa} : \hat{\mathfrak{g}}_\kappa\text{-mod}^{\mathcal{L}^+G}_{\mathrm{Ran}} \longrightarrow D_\kappa(\mathrm{Bun}_G)$$

This is given by Ind-Coh pull-push along:

$$\begin{array}{ccc} & \left((\widehat{\mathcal{L}G})_{\mathcal{L}^+G}\backslash \mathrm{Bun}_G^{\mathrm{lev}}\right)_{\mathrm{Ran}_{dR}} & \\ \swarrow & & \searrow \\ \mathbb{B}_{\mathrm{Ran}_{dR}}\left(\widehat{\mathcal{L}G}_{\mathcal{L}^+G}\right) & & (\mathrm{Bun}_G)_{dR} \end{array}$$

de Rham stack of Ran

classifying stack

complete along arc space

(w/ κ-twist + inf. dim subtleties).

Prop $\mathrm{Loc}_{G,\kappa}(A)\big|^!_{p^{\mathrm{triv}}} \simeq C_\nabla(X, \mathrm{oblv}_{(\hat{\mathfrak{g}}_\kappa,\, \mathcal{L}^+G)} A)$

$\forall A \in \mathrm{FactAlg}(\hat{\mathfrak{g}}_\kappa\text{-mod}^{\mathcal{L}^+G}_{\mathrm{Ran}})$

May twist A "by $\mathcal{P}$" to describe any fiber $\mathcal{P} \in \mathrm{Bun}_G(k)$ of Loc.

Prop $A \in \mathrm{FactAlg}(\hat{\mathfrak{g}}_\kappa\text{-mod}^{\mathcal{L}^+G}_{\mathrm{Ran}})$, $B \in \mathrm{Com\,FactAlg}$.

KL-category

Then $\mathrm{Loc}_{G,k}(B \overset{!}{\otimes} A) \simeq C_\nabla(X, B) \otimes \mathrm{Loc}_{G,k}(A)$

<u>Pf</u> Very similar to proof that $C_\nabla(X, -)$ is symm. monoidal ∎

$A \in \mathrm{FactAlg}(\hat{\mathfrak{g}}_k\text{-mod}^{\mathfrak{L}^+G}_{\mathrm{Ran}}) \rightsquigarrow Z(A) := \mathrm{End}_{A\text{-mod}^{\mathrm{Fact}}(\hat{\mathfrak{g}}_k\text{-mod}^{\mathfrak{L}^+G}_{\mathrm{Ran}}), D(\mathrm{Ran})}(\mathrm{Vac}_A)$
$\in \mathrm{AssocAlg}(\mathrm{Fact\,Alg})$

B = comm fac. alg, $B \to Z(A)$ in $\mathrm{Assoc\,Alg}(\mathrm{Fact\,Alg})$

$\rightsquigarrow C_\nabla(X, B) \curvearrowright \mathrm{Loc}_{G,k}(A) \in D_k(\mathrm{Bun}_G)$.

$A = A_{\mathfrak{g},\mathrm{crit}}$, $B = \mathrm{Fun}(\mathrm{Op}_{\check{G}}(D))$

$\rightsquigarrow \mathrm{Fun}(\mathrm{Op}_{\check{G}}(X)) \curvearrowright D_{\mathrm{Bun}_G, \mathrm{crit}} = \mathrm{Loc}_{G,\mathrm{crit}}(A_{\mathfrak{g},\mathrm{crit}})$,

accomplishes 1st desired property.

§ Factorizable Satake

$\mathrm{Rep}\, G$ symm. mon. cat $\rightsquigarrow$ fact. cat

A geometric construction:

$$\mathrm{Map}(S, LS_G(D)_{X^I_{dR}}) := \left\{ (x_I, E_G) \;\middle|\; \begin{array}{l} x_I : S \to X^I_{dR} \\ (D_{x_I})_{dR} \times_{S_{dR}} S \xrightarrow{E_G} pt/G \end{array} \right\}$$

$LS_G(D)_x = pt/G$

$LS_G(D)_{X^I}$ is <u>not</u> locally of fin. type (despite fibers and base being lft!)

$\mathrm{Rep}(G)_{X^I} := \mathrm{QCoh}(LS_G(D)_{X^I_{dR}})$

$\mathrm{Rep}(G)_{\mathrm{Ran}} = \lim_I \mathrm{Rep}(G)_{X^I} = \mathrm{QCoh}(LS_G(D)_{\mathrm{Ran}_{dR}})$

$\mathrm{Loc}_{G,\infty} : \mathrm{Rep}(G)_{\mathrm{Ran}} \longrightarrow \mathrm{QCoh}(LS_G(X))$ pull-push along

$$\begin{tikzcd} & \mathrm{Ran}_{dR} \times LS_G(X) \arrow[dl] \arrow[dr] & \\ LS_G(D)_{\mathrm{Ran}_{dR}} & & LS_G(X) \end{tikzcd}$$

Prop (Gaitsgory–Lurie) $\mathrm{Loc}_{G,\infty}$ admits a cts & fully faithful right adjoint.

pf Notes from Jerusalem '14 (local to global)

Prop $\mathrm{Loc}_{G,\infty}$ is symm mon on FactAlg $(\mathrm{Rep}(G)_{\mathrm{Ran}})$.

pf Similar to pf that $C_\nabla(X,-)$ is symm monoidal ∎

$B = \mathrm{ComAlg}(D(X)) \simeq \mathrm{Com\,FactAlg}$

Suppose we are given $\mathrm{Spec}\,B \longrightarrow pt/G$

$(\Longleftrightarrow$ G-torsor $\mathrm{Spec}\,B^{enh} \longrightarrow \mathrm{Spec}\,B)$

$B^{enh} \in \mathrm{ComAlg}(\mathrm{Rep}\,G_X)$, $(B^{enh})^G = B$.

Prop $R\Gamma(LS_G(X), \mathrm{Loc}_{G,\infty}(B^{enh})) \simeq C_\nabla(X, B)$

PF $\mathrm{Spec}\,B \longrightarrow LS_G(D)_{X_{dR}} = X_{dR} \times pt/G$

$\overset{\mathrm{Sect}_\nabla(X,-)}{\rightsquigarrow}$ $\mathrm{Spec}\,C_\nabla(X,B) \longrightarrow LS_G(X)$. ∎

Thm ∃ canonical monoidal functor

$$\mathrm{Rep}(\check{G})_{\mathrm{Ran}} \longrightarrow (\mathrm{Sph}_{G,\mathrm{crit}})_{\mathrm{Ran}}$$

compatible w/ factorization.

pf Raskin CPSI

$$\mathrm{Rep}(\check{G})_{\mathrm{Ran}} \longrightarrow (\mathrm{Sph}_{G,\mathrm{crit}})_{\mathrm{Ran}} \curvearrowright \hat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}^{\mathcal{L}^+G}_{\mathrm{Ran}}$$

$$A \in \mathrm{FactAlg}(\hat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}^{\mathcal{L}^+G}_{\mathrm{Ran}})$$

$$\rightsquigarrow Z^{\mathrm{enh}}(A) := \underline{\mathrm{End}}_{A\text{-mod}^{\mathrm{fact}}(\hat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}^{\mathcal{L}^+G}_{\mathrm{Ran}}),\, \mathrm{Rep}(\check{G})_{\mathrm{Ran}}}(\mathrm{Vac}_A)$$

$$\in \mathrm{AssocAlg}(\mathrm{FactAlg}(\mathrm{Rep}(\check{G})_{\mathrm{Ran}}))$$

$$Z^{\mathrm{enh}}(A)^{G^\vee} = Z(A)$$

$\underline{\text{Thm}}$ $\mathrm{Rep}(\check{G})_{\mathrm{Ran}} \longrightarrow (\mathrm{Sph}_{G,\mathrm{crit}})_{\mathrm{Ran}} \curvearrowright D_{\mathrm{crit}}(\mathrm{Bun}_G)$

(Gaitsgory, vanishing...)

factors through $\mathrm{Loc}_{\check{G},\infty} : \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \longrightarrow \mathrm{QCoh}(LS_{\check{G}}(X))$.

B, B^{enh} as before. $B^{\mathrm{enh}} \longrightarrow Z^{\mathrm{enh}}(A)$ in $\mathrm{AssocAlg}(\mathrm{FactAlg}(\mathrm{Rep}\,\check{G}_{\mathrm{Ran}}))$.

$$\rightsquigarrow \mathrm{Loc}_{\check{G},\infty}(B^{\mathrm{enh}}) \longrightarrow \underline{\mathrm{End}}_{D^{\mathrm{crit}}(\mathrm{Bun}_G),\, \mathrm{QCoh}\, LS_{\check{G}}(X)}(\mathrm{Loc}_{G,\mathrm{crit}}(A))$$

in $\mathrm{AssocAlg}\ \mathrm{QCoh}\ LS_{G^\vee} X$.

Equivalently, $C_\bullet(X,B)\text{-mod} \longrightarrow D_{\mathrm{crit}}(\mathrm{Bun}_G)$, equivariant for $\mathrm{QCoh}\, LS_{\check{G}}(X)$.

$C_\bullet$ '' conformal blocks

$$C_\bullet(X,B) \mapsto D_{\mathrm{crit}}$$

Our case: $A = A_{\mathfrak{g},\mathrm{crit}}$, $B = \mathrm{Fun}(\mathrm{Op}_{\check{G}}(D))$

$$\rightsquigarrow \mathrm{QCoh}\,\mathrm{Op}_{\check G} X \longrightarrow D_{crit}(\mathrm{Bun}_G) \quad \text{equiv. for } \mathrm{QCoh}\, LS_{\check G} X.$$

$$\mathcal{O} \longmapsto \mathcal{D}$$

$\Rightarrow$ Hecke eigenvalue property

§ FLE at critical level

$$\mathrm{Op}_G^{unr, loc} := \mathrm{Op}_G(\mathring{D}) \times_{LS_G(\mathring{D})} LS_G(D)$$ factorization space.

(lives over Ran_{dR})

<u>Thm</u> (FLE at crit. level) $\exists$ canonical equiv of fact. cat's

$$\hat{\mathfrak{g}}_{crit}\text{-mod}^{L^+G}_{Ran} \xrightarrow{\sim} \mathrm{IndCoh}(\mathrm{Op}_{\check G}^{unr, loc})_{Ran} \quad \text{equiv. for } \mathrm{Rep}\,\check G_{Ran}.$$

Drinfeld Sokolov reduction. Ref: Chen - Gaitsgory - Raskin.

Supposed to have

$$\begin{array}{ccc}
\hat{\mathfrak{g}}_{crit}\text{-mod}^{L^+G}_{Ran} & \xrightarrow[\;G\;]{FLE_{crit},\ \sim} & \mathrm{ICoh}(\mathrm{Op}_{\check G}^{unr, loc})_{Ran} \\
\downarrow {\scriptstyle Loc_{G,crit}} & & \downarrow \\
D_{crit}(\mathrm{Bun}_G) & \xrightarrow[\sim]{\mathbb{L}_G} & \mathrm{ICoh}_{\mathcal{N}}(LS_{\check G} X)
\end{array}$$

The right vertical functor is pull-push along

$$\mathrm{Op}_{\check G}^{unr, glob} := \mathrm{Op}_{\check G}(X \setminus x_{univ}) \times_{LS_{\check G}(X\setminus x_{univ})} LS_{\check G}(X)$$

$$\mathrm{Op}_{\check G}^{unr, loc} \longleftarrow \mathrm{Op}_{\check G}^{unr, glob} \longrightarrow LS_{\check G} X$$

$$
\begin{tikzcd}
\mathrm{IndCoh}(\mathrm{Op}^{\mathrm{unr,loc}}_{\check{G}})_{\mathrm{Ran}} \arrow[r, "\mathrm{FLE}^{-1}_{\mathrm{crit}}", "\sim"'] \arrow[dr] & \hat{\mathfrak{g}}_{\mathrm{crit}}\text{-}\mathrm{mod}^{L^+G}_{\mathrm{Ran}} \arrow[d] \\
\mathrm{Rep}\,\check{G}_{\mathrm{Ran}} \circlearrowright \arrow[dr] & D_{\mathrm{crit}}(\mathrm{Bun}_G) \\
 & \circlearrowright \\
 & \mathrm{QCoh}\,\mathrm{LS}_{\check{G}}\,X
\end{tikzcd}
$$

$$\rightsquigarrow \mathrm{IndCoh}(\mathrm{Op}^{\mathrm{unr,loc}}_{\check{G}})_{\mathrm{Ran}} \underset{\mathrm{Rep}\,\check{G}_{\mathrm{Ran}}}{\otimes} \mathrm{QCoh}(\mathrm{Ran}_{dR} \times \mathrm{LS}_{\check{G}}\,X) \longrightarrow D_{\mathrm{crit}}^{\mathrm{Bun}_G}$$

This is slightly dubious. $\rightarrow$ is

$$\mathrm{IndCoh}(\mathrm{Op}^{\mathrm{unr,glob}}_{\check{G}})_{\mathrm{Ran}} \qquad \mathrm{QCoh}\,\mathrm{LS}_{\check{G}}\,X\text{-equiv.}$$